Scrambling Time from Local Perturbations of the Rotating BTZ Black Hole

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Swansea Theory Group Seminar

2016-10-21

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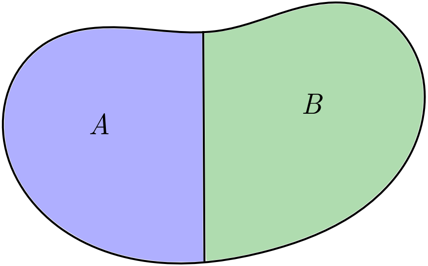
arXiv:1503.08161
AS, to appear

Open systems

System is divided into lab A and environment B

$$|\psi\rangle_{AB} \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$



Measurement in the lab: $\mathcal{O} = \mathcal{O}_A \otimes \mathbb{I}_B$

Density matrices

Example: 2 qubit system

$$|\psi\rangle_{AB} = p\,|0\rangle_A\,|0\rangle_B + q\,|1\rangle_A\,|1\rangle_B$$
$$\langle \mathcal{O}_A \rangle = |p|^2 \langle 0|\mathcal{O}_A|0\rangle + |q|^2 \langle 1|\mathcal{O}_A|1\rangle$$

Can not be written as $\langle \mathcal{O}_A \rangle = \langle \phi|\mathcal{O}_A|\phi\rangle$

Expectation value

$$\langle \mathcal{O} \rangle = \mathrm{Tr}_{\mathcal{H}_A}\, \mathcal{O}_A \rho_A$$

Density matrix

$$\rho_A = |p|^2\,|0\rangle\langle 0| + |q|^2\,|1\rangle\langle 1|$$

Density matrices

General bipartite system

$$|\psi\rangle_{AB} = \sum_{i,j} p_{ij} |i\rangle_A |j\rangle_B$$

Density matrix

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}(|\psi\rangle_{AB} \langle\psi|_{AB})$$

Thermal density matrix

Thermal states obey **Boltzmann** distribution

$$p_n = \frac{1}{Z(\beta)} e^{-\beta E_n}$$

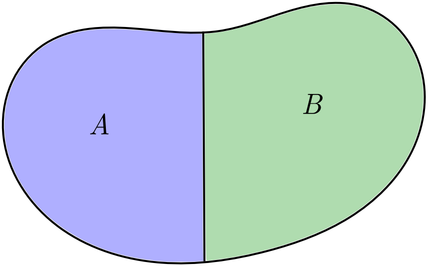
Thermal density matrix

$$\rho = \frac{1}{Z(\beta)} \sum_n e^{-\beta E_n} |\Psi_n\rangle \langle\Psi_n| = \frac{1}{Z(\beta)} e^{-\beta H}$$

Entanglement entropy

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Trace over Hilbert space of environment $\to \rho_A$



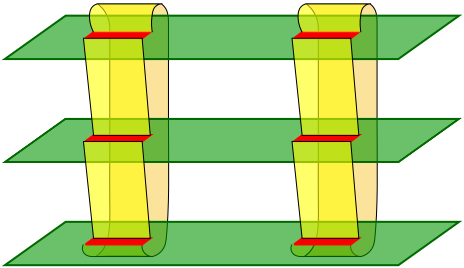
Entanglement entropy is von Neumann entropy of reduced density matrix

Entanglement entropy

$$S_A = -\operatorname{Tr}_A \left(\rho_A \log \rho_A\right)$$

Replica trick

- Calculate $S_n(A) = \dfrac{1}{1-n} \log \mathrm{Tr}_A(\rho_A^n)$ on $n-$sheeted Riemann surface

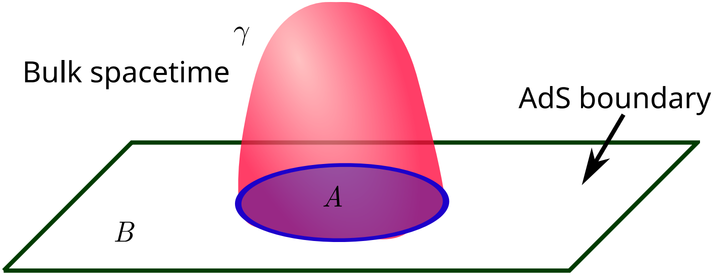


- Continue analytically
- $S_A = \lim_{n\to 1} S_n(A)$

Holographic entanglement entropy

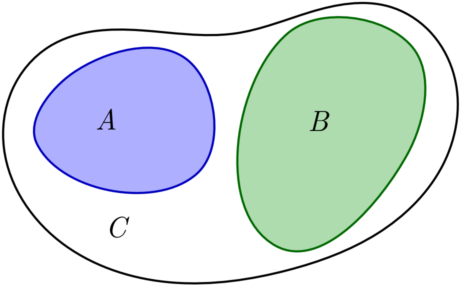
Ryu-Takayanagi formula

$$S_A = \min_{\partial\gamma=\partial A} \frac{\text{Area}(\gamma)}{4G_N}$$



Mutual information

Entanglement between two systems A and B



Mutual information

$$I_{A:B} = S_A + S_B - S_{A\cup B} \geq \frac{\left(\langle \mathcal{O}_A \mathcal{O}_B \rangle - \langle \mathcal{O}_A \rangle \langle \mathcal{O}_B \rangle\right)^2}{2\|\mathcal{O}_A\|^2 \|\mathcal{O}_B\|^2}$$

Provides an upper bound on correlators between A and B

Scrambling time

A natural question to ask is when correlators and entanglement get destroyed by the perturbation?

Scrambling time is defined by

$$I_{A:B}(t^{\star}) = 0$$

Our goal is to compute the **scrambling time** for thermofield double state

Thermofield double state (TFD)

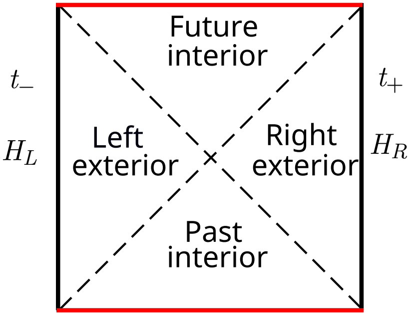
Two non-interacting 2d CFTs with Hilbert spaces $\mathcal{H}_L \cong \mathcal{H}_R$
Choose a particular **entangled** state

Thermofield double state in $\mathcal{H}_L \otimes \mathcal{H}_R$

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta}{2} E_n} |n\rangle_L |n\rangle_R$$

Tracing over $\mathcal{H}_L$ or $\mathcal{H}_R$ yields *thermal* reduced density matrix

Gravity dual description: Eternal BTZ



Two natural choices for Hamiltonian:

- $H = H_L \otimes \mathbb{I}_L - \mathbb{I}_R \otimes H_R$. $t_- = t_+ = t$
- $H = H_L \otimes \mathbb{I}_L + \mathbb{I}_R \otimes H_R$. $t_- = -t_+ = t$. Approximate description of two AdS black holes

Adding a local perturbation

Perturb TFD by a primary operator ψ in the CFT_L at time $-t_\omega$ in the past

- $\rho_{\text{thermal}} = e^{\beta H_L}$
- Perturbation in the left CFT $\psi(0,-t_\omega)e^{\beta H_L}\psi^\dagger(0,-t_\omega)$
- Energy of perturbation is infinite, smear perturbation

$$\psi(0,-t_\omega) \mapsto e^{-\epsilon H_L}\psi(0,-t_\omega)e^{\epsilon H_L}$$

- Evolve in time $e^{-iH_L t}\rho(0)e^{iH_L t}$

$$\rho_L(t) \sim e^{-iH_L t}e^{-\epsilon H_L}\psi(0,-t_\omega)e^{\epsilon H_L+\beta H_L+\epsilon H_L}\psi^\dagger(0,-t_\omega)e^{-\epsilon H_L}e^{iH_L t}$$

Perturbation in Euclidean time

Simpler in Euclidean time $(x + i\tau, x - i\tau)$

$$\rho_L(t) \sim \psi(x_2, \bar{x}_2) e^{\beta H_L} \psi^\dagger(x_1, \bar{x}_1)$$

where

$$x_1 = t_- + t_\omega + i\epsilon, \qquad x_2 = t_- + t_\omega - i\epsilon$$
$$\bar{x}_1 = -t_- - t_\omega - i\epsilon, \qquad \bar{x}_2 = -t_- - t_\omega + i\epsilon$$

Thermal state $\Rightarrow \tau \sim \tau + \beta$

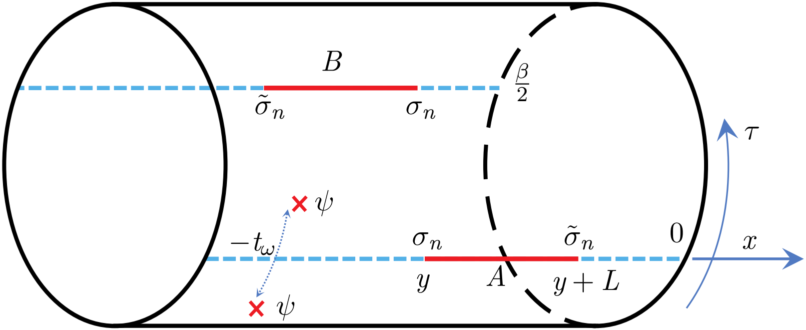
Analytic continuation in TFD

$$\langle \Psi_\beta | \mathcal{O}_L(x_1, 0) \mathcal{O}_L(x_2, t) | \Psi_\beta \rangle = \langle \Psi_\beta | \mathcal{O}_L(x_1, 0) \mathcal{O}_R^\dagger\left(x_2, t - i\frac{\beta}{2}\right) | \Psi_\beta \rangle$$

One-sided and two-sided correlators in TFD are related

$$t \mapsto t + i\frac{\beta}{2}$$

Setup for mutual information



Replica trick for S_A

Replica trick $\rightarrow$ replicate cylinder n times
Glue along the cuts
Instead, twist operators can be used

$$\mathrm{Tr}\,\rho_A^n(t) = \frac{\langle \Psi(x_1,\bar{x}_1)\sigma_n(x_2,\bar{x}_2)\tilde{\sigma}_n(x_3,\bar{x}_3)\Psi^\dagger(x_4,\bar{x}_4)\rangle_{C_n}}{\left(\langle \psi(x_1,\bar{x}_1)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_1}\right)^n}$$

$$\begin{aligned} x_1 &= -i\epsilon, \quad & x_2 &= y - t_\omega - t_-, \quad & x_3 &= y + L - t_\omega - t_-, \quad & x_4 &= +i\epsilon \\ \bar{x}_1 &= +i\epsilon, \quad & \bar{x}_2 &= y + t_\omega + t_-, \quad & \bar{x}_3 &= y + L + t_\omega + t_-, \quad & \bar{x}_4 &= -i\epsilon \end{aligned}$$

$$\Psi = \psi_1 \cdot \psi_2 \cdot \ldots \cdot \psi_n$$

Ψ has a ψ_i for each copy of the theory
Conformal dimension nh_ψ
Conformal dimension of twist operators is $2H_\sigma$

$$H_\sigma = \frac{c}{24}\left(n - \frac{1}{n}\right)$$

Correlators on the plane

We are in a thermal state on the cylinder

Map to the plane

$$w(x) = e^{\frac{2\pi}{\beta_+}x} \qquad \bar{w}(\bar{x}) = e^{\frac{2\pi}{\beta_-}\bar{x}}$$

- 2-pt correlators fixed by conformal symmetry
- 4-pt correlators expressed via conformal blocks $G(z,\bar{z})$

$$\begin{aligned}\operatorname{Tr}\rho_A^n(t) = &\left|\frac{\beta_+}{\pi\varepsilon_{UV}}\sinh\left(\frac{\pi(x_2-x_3)}{\beta_+}\right)\right|^{-2H_\sigma} |1-z|^{4H_\sigma} \\ &\left|\frac{\beta_-}{\pi\varepsilon_{UV}}\sinh\left(\frac{\pi(\bar{x}_2-\bar{x}_3)}{\beta_-}\right)\right|^{-2H_\sigma} G(z,\bar{z})\end{aligned}$$

where

$$z = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_3)(w_2-w_4)} = 1 + f(y,L,\beta_+,t)\epsilon + \mathcal{O}(\epsilon^2)$$

Evaluating $G(z,\bar{z})$

4-pt function $\langle \Psi(x_1,\bar{x}_1)\sigma_n(x_2,\bar{x}_2)\tilde{\sigma}_n(x_3,\bar{x}_3)\Psi^\dagger(x_4,\bar{x}_4)\rangle_{C_n}$
contains two light and two heavy operators
Fitzpatrick, Kaplan and Walters considered such setup in arXiv:1403.6829

$$\log G(z,\bar{z}) \simeq -\frac{c(n-1)}{6}\log \frac{z^{\frac{1}{2}(1-\alpha_\psi)}\bar{z}^{\frac{1}{2}(1-\bar{\alpha}_\psi)}(1-z^{\alpha_\psi})(1-\bar{z}^{\bar{\alpha}_\psi})}{\alpha_\psi\bar{\alpha}_\psi}$$

where

$$\alpha_\psi = \sqrt{1-\frac{24h_\psi}{c}}\,,$$

Cross-ratios

$$z = 1 + \frac{2\pi i\epsilon}{\beta_+} \frac{\sinh \frac{\pi L}{\beta_+}}{\sinh \frac{\pi(y+L-t_- - t_\omega)}{\beta_+} \sinh \frac{\pi(y-t_- - t_\omega)}{\beta_+}} + \mathcal{O}(\epsilon^2)$$

$$\bar{z} = 1 - \frac{2\pi i\epsilon}{\beta_-} \frac{\sinh \frac{\pi L}{\beta_-}}{\sinh \frac{\pi(y+L+t_- + t_\omega)}{\beta_-} \sinh \frac{\pi(y+t_- + t_\omega)}{\beta_-}} + \mathcal{O}(\epsilon^2)$$

$$z \to e^{2\pi i} \quad \text{for} \quad y < t_- + t_\omega < y + L$$

$$\bar{z} \to 1$$

Entanglement entropy S_A

$$S_A = S_{\text{thermal}} + \Delta S_A$$

Thermal part of entanglement entropy is

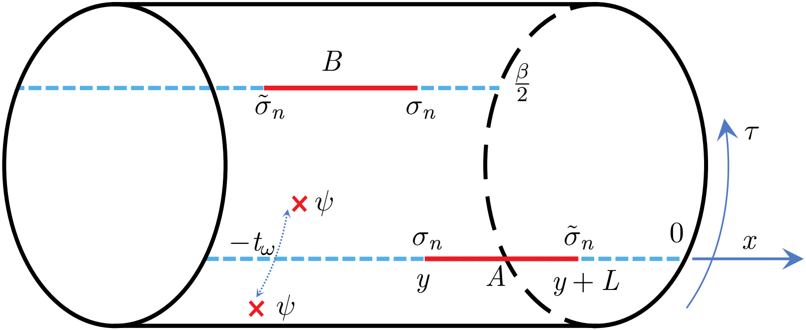
$$S_{\text{thermal}} = \frac{c}{6} \log \left(\frac{\beta_+ \beta_-}{\pi^2 \varepsilon_{UV}^2} \sinh \frac{\pi L}{\beta_+} \sinh \frac{\pi L}{\beta_-} \right)$$

ΔS_A, entanglement entropy due to perturbation

$$\begin{array}{ll} 0, & t_- + t_\omega < y \\ \frac{c}{6} \log \left[\frac{\beta_+}{\pi \epsilon} \frac{\sin \pi \alpha_\psi}{\alpha_\psi} \frac{\sinh \left(\frac{\pi (y + L - t_- - t_\omega)}{\beta_+} \right) \sinh \left(\frac{\pi (t_- + t_\omega - y)}{\beta_+} \right)}{\sinh \left(\frac{\pi L}{\beta_+} \right)} \right] & \\ 0, & t_- + t_\omega > y + L \end{array}$$

No change before perturbations arrives or after it leaves A

Setup for mutual information



Entanglement entropy S_B

$$\mathrm{Tr}\,\rho_B^n(t) = \frac{\langle \Psi(x_1,\bar{x}_1)\sigma_n(x_5,\bar{x}_5)\tilde{\sigma}_n(x_6,\bar{x}_6)\Psi^\dagger(x_4,\bar{x}_4)\rangle_{C_n}}{\left(\langle \psi(x_1,\bar{x}_1)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_1}\right)^n}$$

$$x_5 = y + L + i\frac{\beta}{2} - t_\omega - t_+, \quad x_6 = y + i\frac{\beta}{2} - t_\omega - t_+$$
$$\bar{x}_5 = y + L - i\frac{\beta}{2} + t_\omega + t_+, \quad \bar{x}_6 = y - i\frac{\beta}{2} + t_\omega + t_+$$

Entanglement entropy remains thermal

$$S_B = \frac{c}{6}\log\left(\frac{\beta_+\beta_-}{\pi^2\varepsilon_{UV}^2}\sinh\frac{\pi L}{\beta_+}\sinh\frac{\pi L}{\beta_-}\right)$$

$A \cup B$

$$\mathrm{Tr}\,\rho^n_{A\cup B} = \frac{\langle \psi(x_1,\bar{x}_1)\sigma_n(x_2,\bar{x}_2)\tilde{\sigma}_n(x_3,\bar{x}_3)\sigma_n(x_5,\bar{x}_5)\tilde{\sigma}_n(x_6,\bar{x}_6)\psi^\dagger(x_4,\bar{x}_4)\rangle}{\left(\langle \psi(x_1,\bar{x}_1)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_1}\right)^n}$$

Map to the plane $w(x) = \exp\left(\frac{2\pi}{\beta_+}x\right)$ and

$$z(w) = \frac{(w_1 - w)(w_3 - w_4)}{(w_1 - w_3)(w - w_4)}$$

We obtain the following 6-pt function

$$\langle \psi|\sigma_n(z,\bar{z})\tilde{\sigma}_n(1,1)\sigma_n(z_5,\bar{z}_5)\tilde{\sigma}_n(z_6,\bar{z}_6)|\psi\rangle$$

S and T-channels: Resolution of Identity

S-channel

$$\langle\psi|\sigma_n(z,\bar{z})\tilde{\sigma}_n(1,1)\sigma_n(z_5,\bar{z}_5)\tilde{\sigma}_n(z_6,\bar{z}_6)|\psi\rangle$$
$$=\sum_\alpha \langle\psi|\sigma_n(z,\bar{z})\tilde{\sigma}_n(1,1)|\alpha\rangle\,\langle\alpha|\sigma_n(z_5,\bar{z}_5)\tilde{\sigma}_n(z_6,\bar{z}_6)|\psi\rangle$$

T-channel

$$\langle\psi|\sigma_n(z,\bar{z})\tilde{\sigma}_n(1,1)\sigma_n(z_5,\bar{z}_5)\tilde{\sigma}_n(z_6,\bar{z}_6)|\psi\rangle$$
$$=\sum_\alpha \langle\psi|\sigma_n(z,\bar{z})\tilde{\sigma}_n(z_6,\bar{z}_6)|\alpha\rangle\,\langle\alpha|\sigma_n(z_5,\bar{z}_5)\tilde{\sigma}_n(1,1)|\psi\rangle$$

S-channel

OPE of twist operators

$$\sigma_n(z,\bar{z})\tilde{\sigma}_n(1,1) \sim \mathbb{I} + \mathcal{O}\left((z-1)^r\right) \qquad r \in \mathbb{N}$$

Orthogonality of 2-pt functions

$$\sum_\alpha \langle\psi|\sigma_n(z,\bar{z})\tilde{\sigma}_n(1,1)|\alpha\rangle \langle\alpha| \simeq \langle\psi|\sigma_n(z,\bar{z})\tilde{\sigma}_n(1,1)|\psi\rangle \langle\psi|$$

So the first 4-pt function is the same as in S_A while the second the same as in S_B

$S_{A\cup B} = S_A + S_B$ and $\boxed{I_{A:B} = 0}$

T-channel

Again, dominant contribution comes from $|\alpha\rangle = |\psi\rangle$.

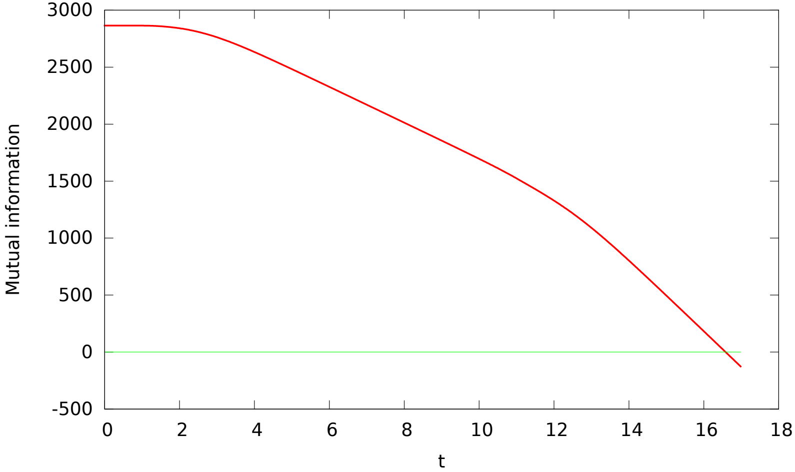
$$\begin{aligned}
\langle\psi|\sigma_n(z_5,\bar{z}_5)\tilde{\sigma}_n(1,1)|\psi\rangle &= G(z_5,\bar{z}_5) \\
\langle\psi|\sigma_n(z,\bar{z})\tilde{\sigma}_n(z_6,\bar{z}_6)|\psi\rangle &= |1-\tilde{z}_2|^{4H_\sigma}|z_2-z_6|^{-4H_\sigma}\,G(\tilde{z}_2,\bar{\tilde{z}}_2)
\end{aligned}$$

Cross-ratios are not the same as in S-channel

Entanglement entropy $S_{A\cup B}$

$$
\begin{aligned}
S_{A\cup B} \simeq \frac{2c}{3}\log\left|\frac{\beta_+\beta_-}{\pi\varepsilon_{UV}}\right| + \frac{c}{3}\log\left(\frac{\beta_+}{\pi\epsilon}\frac{\sin\pi\alpha_\psi}{\alpha_\psi}\right) \\
+ \frac{c}{6}\log\left(\sinh\frac{\pi(t_- + t_\omega - y)}{\beta_+}\cosh\frac{\pi(t_+ + t_\omega - y)}{\beta_+}\right) \\
+ \frac{c}{6}\log\left(\sinh\frac{\pi(t_- + t_\omega - y - L)}{\beta_+}\cosh\frac{\pi(t_+ + t_\omega - y - L)}{\beta_+}\right) \\
t_- + t_\omega > y + L
\end{aligned}
$$

Mutual information



Scrambling time

Scrambling time is defined by

$$I_{A:B}(t_\omega^\star) = 0$$

Suppose $t_\omega > y + L$ and $\dfrac{t_\omega^\star}{\beta} \gg 1$

Scrambling time in perturbed TFD

$$t_\omega^\star = y + \frac{L}{2} - \frac{\beta_+}{2\pi} \log\left(\frac{\beta_+}{\pi\epsilon}\frac{\sin\pi a}{a}\right) + \frac{\beta_+}{\pi}\log\left(2\sinh\frac{\pi L}{\beta_+}\right)$$

Leading term is consistent with scrambling conjecture

$$t_\omega^\star \sim \frac{\beta_+}{2\pi}\log\frac{S}{E}$$

Holographic description

- Gravity dual of TFD is eternal BTZ black hole
- Approximate local perturbation of TFD by a **free falling particle** in the BTZ black hole
- Compute back-reaction of particle on metric of BTZ

Free falling particle

AdS-Schwarzschild patch of the BTZ black hole

$$ds^2 = \frac{R^2}{z^2}\left[-\left(1 - Mz^2\right) dt_-^2 + \frac{dz^2}{1 - Mz^2} + d\theta^2\right], \qquad \theta \sim \theta + 2\pi$$

Add a free falling particle satisfying $z(-t_\omega) = \epsilon$
Energy of the particle

$$E = \frac{m\,R}{\epsilon}\sqrt{1 - M\epsilon^2}$$

matches energy of CFT perturbation if

$$m = \frac{2h_\psi}{R}$$

Back-reaction in AdS_3: Conical defect

Point particle at $r = 0$, $\varphi = 0$ in AdS_3 coordinates

$$ds^2 = -\left(r^2 + R^2\right) d\tau^2 + \frac{R^2\, dr^2}{r^2 + R^2} + r^2\, d\varphi^2$$

Back-reaction in AdS_3 is known:

$$ds^2 = -\left(r^2 + R^2 - \mu\right) d\tau^2 + \frac{R^2\, dr^2}{r^2 + R^2 - \mu} + r^2\, d\varphi^2$$

$\mu = 8\, G_N R^2 m$ is related to mass of the particle m

Back-reaction via embedding coordinates

Coordinates in $\mathrm{AdS}_3 \longleftrightarrow \mathbb{R}^{2,2} \longleftrightarrow \mathrm{BTZ}$ coordinates

$$
\begin{aligned}
\sqrt{R^2+r^2}\sin\tau &= X_0 = R\frac{u+v}{1+uv} \\
\sqrt{R^2+r^2}\cos\tau &= X_1 = R\frac{1-uv}{1+uv}\cosh\phi \\
r\sin\varphi &= X_2 = R\frac{1-uv}{1+uv}\sinh\phi \\
r\cos\varphi &= X_3 = R\frac{u-v}{1+uv}
\end{aligned}
$$

$$
-X_0^2 - X_1^2 + X_2^2 + X_3^2 = -R^2
$$

But $u(v) \not\mapsto r = 0$

Initial conditions: static BTZ

Particle at

$$
\begin{aligned}
z(-t_\omega) &= \epsilon, \quad \theta = 0, \quad t_- = t \\
\dot{z}(-t_\omega) &= 0, \quad \dot{\theta} = 0, \quad \dot{t}_- = 1
\end{aligned}
$$

Map between Schwarzschild and embedding coords

$$
\begin{aligned}
X_0 &= \pm \frac{R\sqrt{1 - Mz^2}}{\sqrt{M}z} \sinh\left(\sqrt{M} t_\mp\right) \\
X_1 &= \frac{R}{\sqrt{M}z} \cosh\left(\sqrt{M}\theta\right) \\
X_2 &= \frac{R}{\sqrt{M}z} \sinh\left(\sqrt{M}\theta\right) \\
X_3 &= \pm \frac{R\sqrt{1 - Mz^2}}{\sqrt{M}z} \cosh\left(\sqrt{M} t_\mp\right)
\end{aligned}
$$

Initial conditions in embedding coordinates

Static BTZ, let $t_\omega = 0$

$$\begin{aligned}
X_0(0) &= 0 \\
X_1(0) &= \frac{R}{\sqrt{M}\epsilon} \\
X_2(0) &= 0 \\
X_3(0) &= \frac{R}{\sqrt{M}\epsilon}\sqrt{1 - M\epsilon^2}
\end{aligned}$$

AdS_3

$$\begin{aligned}
X_0(0) &= 0 \\
X_1(0) &= R \\
X_2(0) &= 0 \\
X_3(0) &= 0
\end{aligned}$$

$$\cosh\lambda_2 = \frac{1}{\sqrt{M}\epsilon}, \qquad \sinh\lambda_2 = \frac{\sqrt{1 - M\epsilon^2}}{\sqrt{M}\epsilon}, \qquad \tanh\lambda_2 = \sqrt{1 - M\epsilon^2}$$

Back-reaction via embedding coordinates

Map to AdS_3 maps dynamic particle to static particle, apply boosts: λ_1 in $X_0 - X_3$ plane and λ_2 in $X_1 - X_3$

$$r \sin\varphi = R\frac{1-uv}{1+uv}\sinh\phi$$

$$r\cos\varphi = \frac{R\cosh\lambda_2(1-uv)}{1+uv}\left(\frac{e^{\lambda_1}u - e^{-\lambda_1}v}{1-uv} - \tanh\lambda_2\cosh\phi\right)$$

Geodesic $\mapsto$ $r = 0$ if

$$\lambda_1 = \sqrt{M}t_\omega, \qquad \tanh\lambda_2 = \sqrt{1-M\epsilon^2}$$

Length of geodesics

Length of *spacelike* geodesics in locally AdS_3 space is known
Let X be region with endpoints $(r_\infty^{(1)}, \tau_\infty^{(1)}, \varphi_\infty^{(1)})$ and $(r_\infty^{(2)}, \tau_\infty^{(2)}, \varphi_\infty^{(2)})$

$$S_X = \frac{c}{6} \log \left[\frac{2 r_\infty^{(1)} r_\infty^{(2)}}{R^2} \frac{\cos\left(|\Delta\tau_\infty| a\right) - \cos\left(|\Delta\varphi_\infty| a\right)}{a^2} \right]$$

where

$$a \equiv \sqrt{1 - \frac{\mu}{R^2}}$$

carries information about perturbation and $c = \frac{3R}{2G_N}$

Propagation of perturbation

Perturbation is inserted at $x = 0$. Region $A = [L_1, L_2]$

- Early time: $t_- + t_\omega < L_1 < L_2$.
  Perturbation has not reached A yet
- Intermediate time: $L_1 < t_- + t_\omega < L_2$
  Perturbation is propagating in A
- Late time: $L_1 < L_2 < t_- + t_\omega$
  Perturbation is outside region A

Geodesics in the left region

Pick region A to be an interval with endpoints

$$(t_-, z_\infty, L_1) \qquad \text{and} \qquad (t_-, z_\infty, L_2)$$

$$r^{(1)} r^{(2)} \simeq \left(\frac{R}{M\epsilon z_\infty}\right)^2 D_1 D_2$$

$$D_i = |\cosh\sqrt{M}L_i - \cosh\sqrt{M}(t_- + t_\omega)|$$

$$\tan\tau^{(i)} \simeq \sqrt{M}\epsilon \frac{\sinh\left(\sqrt{M}(t_- + t_\omega)\right)}{\cosh\left(\sqrt{M}L_i\right) - \cosh\left(\sqrt{M}(t_- + t_\omega)\right)}$$

$$\tan\varphi^{(i)} \simeq \sqrt{M}\epsilon \frac{\sinh\left(\sqrt{M}L_i\right)}{\cosh\left(\sqrt{M}(t_- + t_\omega)\right) - \cosh\left(\sqrt{M}L_i\right)}$$

Geodesics in the left region

- Early time: $t_- + t_\omega < L_1 < L_2$

$$\tau^{(i)} \simeq \sqrt{M}\epsilon \frac{\sinh\left(\sqrt{M}(t_- + t_\omega)\right)}{D_i}$$

$$\varphi^{(i)} \simeq \pi - \sqrt{M}\epsilon \frac{\sinh\left(\sqrt{M}L_i\right)}{D_i}$$

- Late time: $L_1 < L_2 < t_- + t_\omega$

$$\tau^{(i)} \simeq \pi - \sqrt{M}\epsilon \frac{\sinh\left(\sqrt{M}(t_- + t_\omega)\right)}{D_i}$$

$$\varphi^{i} \simeq \sqrt{M}\epsilon \frac{\sinh\left(\sqrt{M}L_i\right)}{D_i}$$

$$S_A \simeq \frac{c}{3}\log\left(\frac{\beta}{\pi z_\infty}\sinh\frac{\pi\Delta L}{\beta}\right) = S_{\text{thermal}}$$

Geodesics in the left region

- Intermediate time: $L_1 < t_- + t_\omega < L_2$

$$\tau^{(1)} \simeq \pi - \sqrt{M}\epsilon \frac{\sinh \sqrt{M}(t_- + t_\omega)}{D_1}$$

$$\tau^{(2)} \simeq \sqrt{M}\epsilon \frac{\sinh \sqrt{M}(t_- + t_\omega)}{D_2}$$

$\Delta\tau$ is no longer $\mathcal{O}(\epsilon)$. Similarly, $\Delta\varphi \approx \pi$.

Entanglement entropy is not thermal

$$\Delta S_A = \frac{c}{6} \log \left[\frac{\beta}{\pi\epsilon} \frac{\sin \pi a}{a} \frac{\sinh\left(\frac{\pi(y+L-t_- -t_\omega)}{\beta}\right) \sinh\left(\frac{\pi(t_- + t_\omega - y)}{\beta}\right)}{\sinh\left(\frac{\pi L}{\beta}\right)} \right]$$

Geodesics in the right region

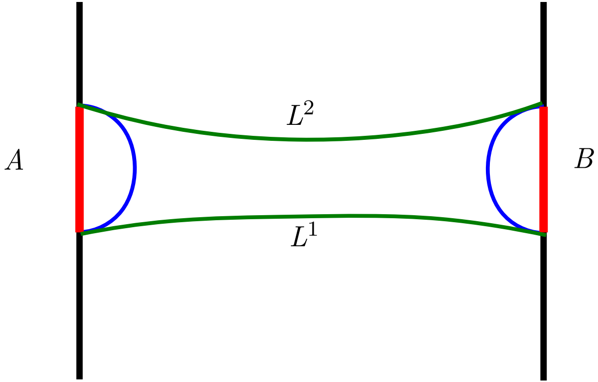
Nothing happens to first order in ϵ
$\Delta\varphi = \mathcal{O}(\epsilon), \quad \Delta\tau = \mathcal{O}(\epsilon)$

$$S_B \simeq S_{\text{thermal}}$$

Entanglement entropy of $A \cup B$

For $A \cup B$ we have two cases

- Endpoints on the same boundary are connected
- Endpoints on different boundaries are connected



Geodesics across the horizon

For $t_- + t_\omega > L_2 > L_1$

$$L_\gamma^1 \simeq \log \left[\left(\frac{\beta \cosh \frac{\pi \Delta t}{\beta}}{\pi z_\infty} \right)^2 \frac{\beta}{\pi \epsilon} \frac{\sin \pi a}{a} \frac{\sinh \frac{\pi(t_- + t_\omega - L_1)}{\beta} \cosh \frac{\pi(L_1 - t_+ - t_\omega)}{\beta}}{\cosh \frac{\pi \Delta t}{\beta}} \right]$$

$$L_\gamma^2 \simeq \log \left[\left(\frac{\beta \cosh \frac{\pi \Delta t}{\beta}}{\pi z_\infty} \right)^2 \frac{\beta}{\pi \epsilon} \frac{\sin \pi a}{a} \frac{\sinh \frac{\pi(t_- + t_\omega - L_2)}{\beta} \cosh \frac{\pi(L_2 - t_+ - t_\omega)}{\beta}}{\cosh \frac{\pi \Delta t}{\beta}} \right]$$

$$S_{A \cup B} = \frac{c}{6}(L_\gamma^1 + L_\gamma^2)$$

Scrambling time

Scrambling time is defined by

$$I_{A:B}(t_\omega^\star) = 0$$

Suppose $t_\omega > L_2$ and $\dfrac{t_\omega^\star}{\beta} \gg 1$

Scrambling time in perturbed TFD

$$t_\omega^\star = \frac{L_1 + L_2}{2} - \frac{\beta}{2\pi} \log\left(\frac{\beta}{\pi\epsilon}\frac{\sin \pi a}{a}\right) + \frac{\beta}{\pi} \log\left(2 \sinh \frac{\pi(L_2 - L_1)}{\beta}\right)$$

Rotating BTZ

Rotating BTZ, let $t_\omega = 0$, $r_\infty = R\epsilon^{-1}$

$$X_0(0) = 0$$

$$X_1(0) = R\sqrt{\frac{r_\infty^2 - r_-^2}{r_+^2 - r_-^2}}$$

$$X_2(0) = 0$$

$$X_3(0) = R\sqrt{\frac{r_\infty^2 - r_+^2}{r_+^2 - r_-^2}}$$

AdS_3

$$X_0(0) = 0$$

$$X_1(0) = R$$

$$X_2(0) = 0$$

$$X_3(0) = 0$$

$$\cosh\lambda_2 = \sqrt{\frac{r_\infty^2 - r_-^2}{r_+^2 - r_-^2}}, \quad \sinh\lambda_2 = \sqrt{\frac{r_\infty^2 - r_+^2}{r_+^2 - r_-^2}}, \quad \tanh\lambda_2 = \sqrt{\frac{R^2 - r_+^2\epsilon^2}{R^2 - r_-^2\epsilon^2}}$$

$$\tanh\lambda_2 = \sqrt{1 - \kappa r_+ \epsilon^2} + \mathcal{O}(\epsilon^4)$$

Idea #1: Diffeomorphism

Rotating BTZ coordinates $\longleftrightarrow \mathbb{R}^{2,2} \longleftrightarrow$ Static BTZ

$$
\begin{aligned}
R\frac{U+V}{1+UV}\cosh\frac{r_-\Phi}{R} - R\frac{V-U}{1+UV}\sinh\frac{r_-\Phi}{R} &= X_0 = R\frac{u+v}{1+uv} \\
R\frac{1-UV}{1+UV}\cosh\frac{r_+\Phi}{R} &= X_1 = R\frac{1-uv}{1+uv}\cosh\frac{r_+\phi}{R} \\
R\frac{1-UV}{1+UV}\sinh\frac{r_+\Phi}{R} &= X_2 = R\frac{1-uv}{1+uv}\sinh\frac{r_+\phi}{R} \\
R\frac{V-U}{1+UV}\cosh\frac{r_-\Phi}{R} - R\frac{V+U}{1+UV}\sinh\frac{r_-\Phi}{R} &= X_3 = R\frac{u-v}{1+uv}
\end{aligned}
$$

$$
u = U\exp\left(\frac{r_-\phi}{R}\right), \quad v = V\exp\left(-\frac{r_-\phi}{R}\right), \quad \phi = \Phi
$$

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R\frac{1-UV}{1+UV}\sinh\frac{r_+\Phi}{R} &= X_2 = R\frac{1-uv}{1+uv}\sinh\frac{r_+\phi}{R} \\
R\frac{V-U}{1+UV}\cosh\frac{r_-\Phi}{R} - R\frac{V+U}{1+UV}\sinh\frac{r_-\Phi}{R} &= X_3 = R\frac{u-v}{1+uv}
\end{aligned}$$

$$u = U\exp\left(\frac{r_-\phi}{R}\right), \quad v = V\exp\left(-\frac{r_-\phi}{R}\right), \quad \phi = \Phi$$

NOT AN ISOMETRY!

Back-reaction map for rotating BTZ

Diffeomorphism

$$u = U \exp\left(\frac{r_- \phi}{R}\right), \quad v = V \exp\left(-\frac{r_- \phi}{R}\right)$$

Can be written as

$$\lambda_1 \mapsto \lambda_1 - \frac{r_- \phi}{R}$$

Length of geodesics in the rotating BTZ

The effect of angle dependent "boost"

- Thermal entropy of static $\mathrm{BTZ} \mapsto$ rotating BTZ
- Change of entropy due to perturbation $\beta \mapsto \beta_+$

Example

In static case $\Delta\tilde{\phi} = 0$

$$S_{\text{thermal}} = \frac{c}{6}\log\left(\frac{\beta^2}{\pi^2 z_\infty^2}\sinh^2\frac{\pi\Delta L}{\beta}\right)$$

Rotating case $\Delta\tilde{\phi} \propto \Delta L$

$$S_{\text{thermal}} = \frac{c}{6}\log\left(\frac{\beta_+\beta_-}{\pi^2 z_\infty^2}\sinh\frac{\pi(\Delta L + \Delta\tilde{\phi})}{\beta}\sinh\frac{\pi(\Delta L - \Delta\tilde{\phi})}{\beta}\right)$$

Summary

1. 2D CFT
   - We perturbed TFD state with primary operator
   - Computed entanglement entropies of two regions
   - Entanglement entropies of union of regions
   - Found *mutual information*
   - Calculated *scrambling time*
2. Gravity
   - Perturbed BTZ with a *falling particle*
   - Mapped to AdS_3
   - Computed *back-reaction*
   - Engtanglement entropy via length of geodesics
   - Diffeomorphism between rotating BTZ and static BTZ